RELATIONS FOR STREAMLINES AND EQUIPOTENTIALS

1. VELOCITY POTENTIAL
   a. Velocity potential $\Phi$ is defined as:
      
      $$\Phi = Kh \quad \text{where} \quad K \text{ is constant}$$
      
      $$\Phi = f(x,y,z)$$

      The units of $\Phi$ are $L^2/T$
   
   b. Then
      
      $$-\nabla \Phi = -K \left( \frac{\partial \Phi}{\partial x} \mathbf{i} + \frac{\partial \Phi}{\partial y} \mathbf{j} + \frac{\partial \Phi}{\partial z} \mathbf{k} \right) = \mathbf{q}$$

      where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors in the x, y, and z directions respectively

      $\mathbf{q}$ is the specific discharge or Darcy velocity vector

      $\nabla \Phi$ has units $L/T$
   
   c. The components in each direction of the specific discharge vector $\mathbf{q}$ are:
      
      $$q_x = -\frac{\partial \Phi}{\partial x} \quad q_y = -\frac{\partial \Phi}{\partial y} \quad q_z = -\frac{\partial \Phi}{\partial z}$$
   
   d. For two dimensions, equipotentials are lines where $\Phi(x,y) = \text{constant}$
   
   e. In two dimensions, the slope of an equipotential is:
      
      $$\left( \frac{dy}{dx} \right)_{\Phi} = -\frac{q_x}{q_y}$$

2. STREAMFUNCTION AND STREAMLINES
   
   a. The streamfunction, $\psi$, is defined as a function which is everywhere tangent to $\mathbf{q}$. Specifically
      
      $$\psi = f(x,y,z)$$

      $\psi$ has units $L^2/T$ (or $L^3/LT$)
   
   b. In two dimensions, streamlines are lines where $\psi(x,y) = \text{constant}$.
   
   c. In two dimensions, the slope of a streamline is:
      
      $$\left( \frac{dy}{dx} \right)_{\psi} = -\frac{q_y}{q_x}$$
   
   d. In two dimensions, the components in each direction of the specific discharge vector $\mathbf{q}$ are:
      
      $$q_x = -\frac{\partial \psi}{\partial y} \quad q_y = -\frac{\partial \psi}{\partial x}$$
   
   e. Between adjacent streamlines, the discharge (per unit of length normal to the x,-y plane), $\Delta Q$, is given by:
      
      $$\Delta Q = \Delta \psi = \psi_2 - \psi_1$$
   
   f. The differential expressions for the components of $\mathbf{q}$ show that streamlines and equipotentials must always be orthogonal in homogeneous isotropic media.
EXAMPLE: DERIVATION OF EQUIPOTENTIALS FROM STREAMFUNCTION

PROBLEM: given \( \psi = x + 2y \), find \( \Phi \)

1. Remember
   \[ q_x = -\frac{\partial \Phi}{\partial x} \quad \text{and} \quad q_y = -\frac{\partial \Phi}{\partial y} \]
   but also
   \[ q_x = -\frac{\partial \psi}{\partial y} \quad \text{and} \quad q_y = -\frac{\partial \psi}{\partial x} \]

   Thus
   \[ \frac{\partial \Phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \text{and} \quad \frac{\partial \Phi}{\partial y} = -\frac{\partial \psi}{\partial x} \]

2. Taking partial differentials of \( \psi \):
   \[ \frac{\partial \psi}{\partial x} = \frac{\partial (x + 2y)}{\partial x} = 1 \quad \frac{\partial \psi}{\partial y} = \frac{\partial (x + 2y)}{\partial y} = 2 \]

   Thus:
   \[ \frac{\partial \Phi}{\partial x} = 2 \quad \frac{\partial \Phi}{\partial y} = -1 \]

3. Integrating each of these expressions with respect to the appropriate variable:
   a. \( \Phi = \int \frac{\partial \Phi}{\partial x} \, dx = \int 2 \, dx = 2x + f(y) \)
   b. \( \Phi = \int \frac{\partial \Phi}{\partial y} \, dy = -1 \int dy = -y + g(x) \)

   \( f(y) \) is added to 3a because \( \Phi \) is a function of both \( x \) and \( y \), and we integrated only with respect to \( x \);
   \( g(x) \) is added to 3b because \( \Phi \) is a function of both \( x \) and \( y \), and we integrated only with respect to \( y \)

4. We need to evaluate \( f(y) \); to do this we differentiate 3b above with respect to \( y \):
   \[ \frac{\partial \Phi}{\partial y} = \frac{\partial (2x + f(y))}{\partial y} = \frac{df}{dy} \]

   But in step 2 above we found that \( \frac{\partial \Phi}{\partial y} = -1 \); hence \( \frac{df}{dy} = -1 \)

   Integrating this to find \( f \):
   \[ f = \int df = - \int dy = -y + \text{constant of integration} \]

5. Thus by substitution into 3a
   \[ \Phi = 2x - y + \text{constant} \]
   The constant is arbitrary and can be set to zero.

6. Thus our final expression defining the equipotentials is:
   \[ \Phi = 2x - y \]